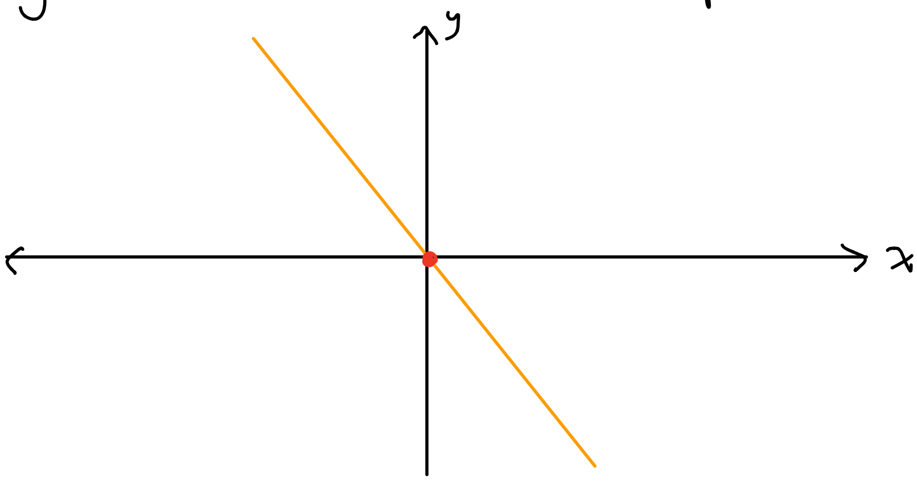
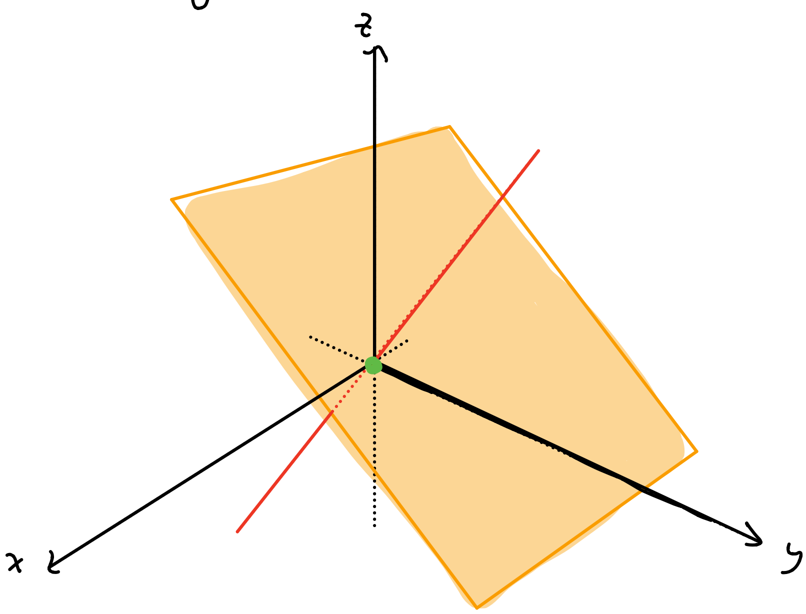
Subspaces of $\mathbb{R}^n$

A subspace of $\mathbb{R}^n$ is a smaller linear space that passes through the origin.

Ex: In $\mathbb{R}^2$, lines through the origin are subspaces. (The origin itself is also a subspace!)



In $\mathbb{R}^3$, the subspaces are planes and lines through the origin, as well as the origin itself.



In general, we define subspaces as follows:

Definition: A set (i.e. collection) U of vectors in $\mathbb{R}^n$ is a <u>subspace</u> of $\mathbb{R}^n$ if it satisfies:

1. The zero vector is in U, i.e. $\vec{0} \in U$

   means "is in"

2. If $\vec{x} \in U$ and $\vec{y} \in U$, then $\vec{x} + \vec{y} \in U$.
   ("closed under addition")

3. If $\vec{x} \in U$, then $a\vec{x} \in U$ for every real number a.
   ("closed under scalar multiplication")

Ex: A plane in $\mathbb{R}^3$ through the origin has equation

$$\vec{n} \cdot \vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0, \quad \text{or} \quad ax + by + cz = 0$$

where $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a normal vector.

Let's check that this is a subspace:

1. If $\vec{v} = \vec{0}$, then $\vec{n} \cdot \vec{v} = \vec{n} \cdot \vec{0} = 0$, so $\vec{0}$ is in the plane.

2. If $\vec{v}_1$ and $\vec{v}_2$ are in the plane, then

$$\vec{n} \cdot (\vec{v}_1 + \vec{v}_2) = \underbrace{\vec{n} \cdot \vec{v}_1}_{0} + \underbrace{\vec{n} \cdot \vec{v}_2}_{0} = 0,$$

so $\vec{v}_1 + \vec{v}_2$ is in the plane.

3. If $\vec{v}$ is in the plane, and s is a scalar, then

$$\vec{n} \cdot (s\vec{v}) = s(\vec{n} \cdot \vec{v}) = s(0) = 0.$$

So any plane through the origin is a subspace of $\mathbb{R}^3$.

Each matrix A has 2 very important subspaces that correspond to it:

Def: Let A be an $m \times n$ matrix.

1. The null space of A, denoted null A, is the set of vectors $\vec{v}$ in $\mathbb{R}^n$ such that $A\vec{v} = \vec{0}$.

$$\text{null } A = \{\vec{v} \in \mathbb{R}^n \mid A\vec{v} = \vec{0}\}.$$

"such that"

2. The image space of A, denoted im$A$, is the set of vectors in $\mathbb{R}^m$ of the form $A\vec{v}$.

That is, vectors $\vec{w}$, such that $A\vec{x} = \vec{w}$ has a solution.

We can check that these are both subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively:

null A:

1. $\vec{0}$ is in null A, since $A\vec{0} = \vec{0}$.

(2.) If $\vec{v}_1$ and $\vec{v}_2$ are in null A, then

$$A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 = \vec{0}.$$

(3.) If $\vec{v} \in$ null A and a is a scalar, then $A(a\vec{v}) = a\underbrace{(A\vec{v})}_{=0} = \vec{0}$.

So null A is a subspace of $\mathbb{R}^m$.

Exercise: Check that im$A$ is a subspace of $\mathbb{R}^n$.

Eigenspaces

Let λ be any number. Define

$$E_\lambda(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\}.$$

Then $\vec{x}$ is in $E_\lambda(A)$ if and only if $(\lambda I - A)\vec{x} = \vec{0}$. So

$$\boxed{E_\lambda(A) = \text{null}\,(\lambda I - A)}$$

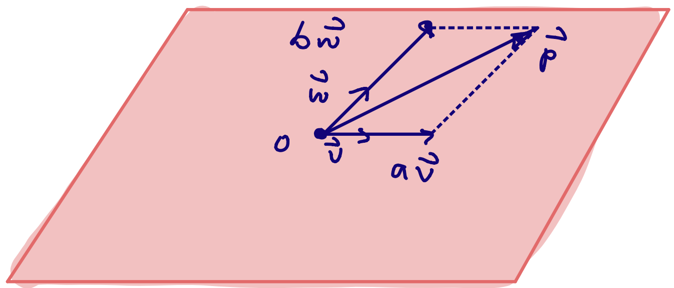
Thus, $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$, called the eigenspace of A corresponding to λ.

If $E_\lambda(A)$ contains any nonzero vectors, then λ is an eigenvalue and the nonzero vectors in $E_\lambda(A)$ are the corresponding eigenvectors.

Spanning sets

Let $\vec{v}$ and $\vec{w}$ be two nonzero, nonparallel vectors in $\mathbb{R}^3$ (w/ tails at the origin). Let M be the plane through the origin containing $\vec{v}$ and $\vec{w}$.

Then a vector $\vec{p}$ is in M if and only if $\vec{p} = a\vec{v} + b\vec{w}$ for some a, b in $\mathbb{R}$, i.e. $\vec{p}$ is a linear combination of $\vec{v}$ and $\vec{w}$.



So we can describe M as

$$M = \{a\vec{v} + b\vec{w} \mid a, b \in \mathbb{R}\}.$$

$\{\vec{v}, \vec{w}\}$ is called a <u>spanning set</u> for M.

More generally:

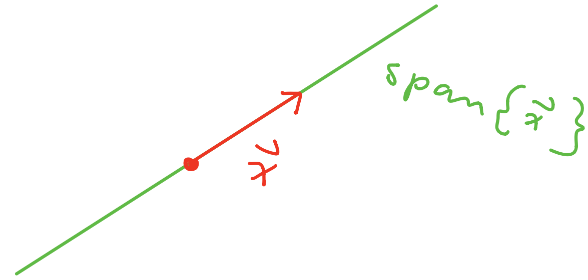
Def: Given $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k$ in $\mathbb{R}^n$, the set of all linear combinations of $\vec{x}_1, \dots, \vec{x}_k$ is called the <u>span</u> of the $\vec{x}_i$, and is denoted

$$\text{span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\} = \{t_1\vec{x}_1 + \dots + t_k\vec{x}_k \mid t_i \text{ in } \mathbb{R}\}.$$

Ex: For any single vector $\vec{x}$, we have

$$\text{span}\{\vec{x}\} = \{t\vec{x} \mid t \in \mathbb{R}\},$$ which is the line through

the origin containing $\vec{x}$.



We saw above that if $\vec{x}, \vec{y}$ are nonparallel vectors, then span$\{\vec{x}, \vec{y}\}$ is the plane through the origin containing $\vec{x}, \vec{y}$.

Thm: Let $U = \text{span}\{\vec{x}_1, \dots, \vec{x}_k\}$ in $\mathbb{R}^n$. Then:

1.) U is a subspace of $\mathbb{R}^n$ containing each $\vec{x}_i$.

2.) If W is a subspace of $\mathbb{R}^n$ and each $x_i \in W$, then $U \subseteq W$ (i.e. U is contained in W).

Ex: If $\vec{e}_1, \dots, \vec{e}_n$ is the standard basis in $\mathbb{R}^n$, then

$$\mathbb{R}^n = \text{span}\{\vec{e}_1, \dots, \vec{e}_n\}$$

since every vector $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ is a linear comb. of the e_i:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_1 \vec{e}_1 + a_2 \vec{e}_2 + \dots + a_n \vec{e}_n.$$

Spanning sets for null A and im A

null A is the set of solutions to $A\vec{x} = \vec{0}$. Thus, if $\vec{v}_1, \dots, \vec{v}_n$ is a set of basic solutions, null A consists of linear combinations of $\vec{v}_1, \ldots, \vec{v}_n$. So

$$\text{null}\, A = \text{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$$

Ex: If $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$, what is null A?

First we solve:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \quad x_1 + 2x_3 = 0, \quad x_2 - x_3 = 0$$

setting $x_3 = s$, $x_4 = t$, solutions are

$$\begin{bmatrix} -2s \\ s \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

so $\text{null}\, A = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

To find im A, let $\vec{c}_1, \ldots, \vec{c}_n$ denote the columns of A. So $A = [\vec{c}_1 \ldots \vec{c}_n]$. Then if $\vec{e}_1, \ldots, \vec{e}_n$ is the standard basis, we have

$$A\vec{e}_i = \vec{c}_i.$$

Thus, each c_i is in im A.

If $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is an arbitrary vector, then

$$\vec{x} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \dots + x_n \vec{e}_n \text{, so}$$

$$A\vec{x} = x_1 A\vec{e}_1 + x_2 A\vec{e}_2 + \dots + x_n A\vec{e}_n$$

$$= x_1 \vec{c}_1 + \dots + x_n \vec{c}_n .$$

So the image of A consists of all linear combinations of the columns of A. i.e.

$$\boxed{\text{im } A = \text{span}\{\vec{c}_1, \dots, \vec{c}_n\}}$$

Ex: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}$, then $\text{im } A = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \right\}$.

Practice Problems: 5.1 : 1, 2b, 3a, 4, 7, 16, 17a